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Connectivity keeping edges in graphs with large minimum degree

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Abstract

The old well-known result of Chartrand, Kaugars and Lick says that every k-connected graph G with minimum degree at least $3k/2$ has a vertex v such that $G - v$ is still k-connected. In this paper, we consider a generalization of the above result [G. Chartrand, A. Kaigars, D.R. Lick, Critically n-connected graphs, Proc. Amer. Math. Soc. 32 (1972) 63–68]. We prove the following result:

Suppose G is a k-connected graph with minimum degree at least $\lfloor 3k/2 \rfloor + 2$. Then G has an edge e such that $G - V(e)$ is still k-connected.

The bound on the minimum degree is essentially best possible.

Keywords: Connectivity; Minimum degree; Edge deletion

1. Introduction

In this paper, all graphs considered are finite, undirected, and without loops or multiple edges. For a graph G, $V(G)$, $E(G)$ and $\delta(G)$ denote the set of vertices and the set of edges and the minimum degree of G, respectively. For $x \in V(G)$, we write $N_G(x)$ for the neighbor-

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hood of $V(G)$ and $d_G(x) = |N_G(x)|$. For a subset S of $V(G)$, the subgraph induced by S is denoted by $\langle S\rangle$. With a slight abuse of notation, for a subgraph H of G and a vertex $v \in V(G)$, $N_H(v) = N_G(v) \cap V(H)$ and $d_H(v) = |N_H(v)|$. In addition, for a subgraph H of G and a subset S of $V(G)$, $N_G(S) = \bigcup_{v\in S} N_G(v) - S$, and when $S \cap V(H) = \emptyset$, $N_H(S) = \bigcup_{v\in S} N_H(v) - S$. Also let $E(A, B)$ be the set of edges between vertex sets A and B in G. When $|A| = 1$, say, $A = \{x\}$, we write $E(x, B)$ as $E(\{x\}, B)$. A subgraph S and its vertex set $V(S)$ are often identified when there is no fear of confusion. Also, for a subgraph X of G, let $E(X)$ denote the set of edges of X. Let $k \geqslant 2$ be an integer. An edge e (resp. triangle T) of a k-connected graph is said to be *k-contractible* if the graph obtained from G by contracting e (resp. T) (and replacing each of the resulting pairs of double edges by a single edge) is still k-connected. Let $E_c(G) = \{e \in E(G) \mid e$ is k-contractible$\}$.

The well known result of Chartrand, Kaugars and Lick [1] is the following.

Theorem 1. *Every k-connected graph G with minimum degree at least $3k/2$ has a vertex v such that $G - v$ is still k-connected.*

A graph G is said to be *critically k-connected* if G is k-connected, but for any vertex v in G, $G - v$ is not k-connected. So, Theorem 1 tells us that every critical k-connected graph has a vertex of degree less than $3k/2$. Mader [7] gave a simpler proof of Theorem 1. Hamidoune [5] generalized this result as follows: Every critically k-connected graph has at least two vertices of degree less than $3k/2$. This result was further extended in [6].

The notion "critically k-connected graph" was generalized as follows. A k-connected graph G is called *l-critically k-connected* if for all vertex set V' with $|V'| \leqslant l \leqslant k$, $G - V'$ is not $(n - |V'| + 1)$-connected. This concept was introduced by Maurer and Slater [12]. Note that 1-critically k-connected graphs are exactly critically k-connected graphs. This concept is paid attention by many researchers, cf. [6,8,9,12]. See Mader's survey [10].

In this paper, we consider a different direction. Theorem 1 tells us that if we want to find a vertex v in a k-connected graph G such that $G - v$ is still k-connected, then the minimum degree $3k/2$ is enough. But what if we want to find an edge e such that $G - V(e)$ is still k-connected? What minimum degree condition is necessary? Motivated by this question, we shall prove the following result.

Theorem 2. *Let G be a k-connected graph with $\delta(G) \geqslant \lfloor 3k/2\rfloor + 2$. Then G has an edge e such that $G - V(e)$ is still k-connected.*

Actually, we shall prove a somewhat stronger result.

The following is our main result, which would immediately imply Theorem 2.

Theorem 3. *Let G be a k-connected graph with $\delta(G) \geqslant \lfloor 3k/2\rfloor + 1$. Then one of the following holds*:

(i) *G has an edge e such that $G - V(e)$ is still k-connected.*
(ii) *G contains a subgraph X such that $X \cong K_{\lfloor k/2\rfloor+1} + (k+1)K_1$ and $E(X) \subset E_c(G)$. Moreover, there is a vertex in X that has degree exactly $\lfloor 3k/2\rfloor + 1$ in G.*

In Theorem 3, the bound on $\delta(G)$ is best possible. To see this, we give the following examples.

Case 1: k is even.

Let X_i be a complete graph of order $k/2$ for each $1 \leqslant i \leqslant m$, and let Y_i be a complete graph of order $k/2 + 1$ for each $1 \leqslant i \leqslant m$. For graphs X, Y, "$X + Y$" means joining each vertex of

X to all vertices of Y completely. Under this notation, consider the graph $G = X_1 + Y_1 + X_2 + Y_2 + \cdots + X_m + Y_m + X_1$.

Case 2: k is odd.

Let X_i be a complete graph of order $(k-1)/2$ for each $1 \leqslant i \leqslant m$, and let Y_i be a complete graph of order $(k+1)/2$ for each $1 \leqslant i \leqslant m$.

As in Case 1, put $G' = X_1 + Y_1 + X_2 + Y_2 + \cdots + X_m + Y_m + X_1$. Let v be a new vertex. Join v to every vertex in Y_i for each i with $1 \leqslant i \leqslant m$ in G'. Let G be a resulting graph.

In both cases, G is a k-connected graph with $\delta(G) = \lfloor 3k/2 \rfloor$. It is easy to check that G does not contain an edge whose deletion results in still k-connected nor $K_{\lfloor k/2 \rfloor + 1} + (k+1)K_1$.

Motivated by Theorems 1 and 3, we conjecture the following.

Conjecture 1. *For fixed l, there is a function $f(l)$ satisfying the following: Suppose G is k-connected with minimum degree at least $3k/2 + f(l)$. Then G has a connected subgraph W of order l such that $G - W$ is still k-connected.*

Theorem 1 implies that $f(1) = 0$. Our result, Theorem 2 implies that $f(2) \leqslant 2$. A similar construction of graphs described above (Cases 1 and 2) shows $f(l) \geqslant l - 1$ (by just replacing Y_i by the corresponding graph depending on l). But we do not know if the value $l - 1$ is best possible.

There are some related conjectures we should mention here. In [11], Mader has conjectured that for all positive integers k and l, there is a least non-negative integer $h(k,l)$ such that every k-connected graph G with order strictly greater than $h(k,l)$ contains a connected subset S of the vertices where the cardinality of S is l and such that the vertex-connectivity number of $G - S$ is at least $k - 3$. In the same paper, Mader has established that every k-connected graph G of sufficiently large order contains a connected graph H on 4 vertices such that $G - V(H)$ is $(k-3)$-connected. In [13], McCuaig and Ota conjectured that for all positive integer l, there is a least non-negative integer $h(l)$ such that every 3-connected graph with at least $h(l)$ vertices has a connected subgraph W of order exactly l such that $G - W$ is 2-connected.

2. Proof of Theorem 3

Let G be a k-connected graph with $\delta(G) \geqslant \lfloor 3k/2 \rfloor + 1$. By contradiction, suppose that G is a counterexample in the theorem. The result of Egawa [3,4] says that every k-connected graph with minimum degree at least $5k/4$ has a k-contractible edge. (This also follows from [2,14].) Hence G has a k-contractible edge.

It is clear that every contractible edge is contained in a $(k+1)$-cutset. Let $A_1 := \{S \mid S$ is a k-cutset which contains an edge$\}$ and $A_2 := \{S \mid S$ is a $(k+1)$-cutset which contains an k-contractible edge$\}$. Note that for every edge $e \in E(G)$, there exists a cutset $S \in A_1 \cup A_2$ such that S contains e.

Lemma 1. *Let S be a cutset in $A_1 \cup A_2$ and let A be a component of $G - S$.*
Then following statements holds:

(i) *If $S \in A_1$, then $|A| \geqslant (k+3)/2$.*
(ii) *If $S \in A_2$, then $|A| \geqslant (k+1)/2$.*

Proof. Since the minimum degree of G is at least $\lfloor 3k/2 \rfloor + 1$, it is obvious. □

Let $Q \in A_1 \cup A_2$ and let H be a component in $G - Q$. Let $W = G - Q - H$. We may assume that A is chosen so that $|H|$ is minimum.

Take $Q' \in A_1 \cup A_2$ so that Q' contains an edge in $E(H) \cup E(H, Q)$.

In the rest of the proof, we use the following notation:

Let H', W' be unions of components of $G - Q'$ such that $H' \neq \emptyset$, $W' \neq \emptyset$ and $V(G) = V(H') \cup V(Q') \cup V(W')$. Let H_1, H_2 and H_3 denote $H \cap H'$, $H \cap Q'$ and $H \cap W'$, respectively. Also, let W_1, W_2 and W_3 denote $W \cap H'$, $W \cap Q'$ and $W \cap W'$, respectively. Let Q_1, Q_2 and Q_3 denote $Q \cap H'$, $Q \cap Q'$ and $Q \cap W'$, respectively. By the choice of Q', it follows that $H_2 \neq \emptyset$.

In view of Lemma 1 and the minimality of $|H|$, note that $|H| \geqslant (k+1)/2$, $|H'| \geqslant (k+1)/2$, $|W| \geqslant (k+1)/2$, $|W'| \geqslant (k+1)/2$.

Lemma 2. *The following two statements hold*:

(i) *If $H_1 \neq \emptyset$, then $W_3 \neq \emptyset$.*
(ii) *If $H_3 \neq \emptyset$, then $W_1 \neq \emptyset$.*

Proof. Suppose that $H_1 \neq \emptyset$ and $W_3 = \emptyset$. Then by the minimality of $|H|$, we see that $|H_1 \cup H_2| \leqslant |Q_3|$. Since $H_1 \neq \emptyset$, we have $|H_2| < |Q_3|$. On the other hand, again by the minimality of $|H|$, we have $|H_2 \cup Q_1 \cup Q_2| \geqslant k+1$. Consequently, $|H_2| + (k+1) - |Q_3| \geqslant |H_2| + |Q - Q_3| \geqslant k+1$, and hence $|H_2| \geqslant |Q_3|$. This is a contradiction. Thus (i) was proved. We can similarly prove (ii). □

Lemma 3. $Q, Q' \in A_2$.

Proof. First we claim that $Q \in A_2$. By contradiction, assume for a while that $Q \in A_1$. Then, by Lemma 1, note that $|H| \geqslant (k+3)/2$. Now we claim that $H_1 = H_3 = \emptyset$. Suppose that $H_1 \neq \emptyset$. Then by the minimality of $|H|$, we see that $|H_2 \cup Q_1 \cup Q_2| \geqslant k+1$ or $|H_2 \cup Q_1 \cup Q_2| \geqslant k+2$ according as $Q' \in A_1$ or $Q' \in A_2$. This implies $|Q_2 \cup Q_3 \cup W_2| \leqslant k-1$. Hence $W_3 = \emptyset$, which contradicts Lemma 2(i). Thus we have $H_1 = \emptyset$. We can similarly obtain $H_3 = \emptyset$ from Lemma 2(ii). Thus $H_1 = H_3 = \emptyset$, as claimed. This implies $|H_2| = |H| \geqslant (k+3)/2$, and hence $|Q_2 \cup W_2| \leqslant (k-1)/2$. Since $|Q_1| + |Q_3| \leqslant k$, we have $|Q_1| \leqslant k/2$ or $|Q_3| \leqslant k/2$. By symmetry, we may assume $|Q_1| \leqslant k/2$. Then $|Q_1 \cup Q_2 \cup W_2| \leqslant k/2 + (k-1)/2 < k$, which implies $W_1 = \emptyset$. Therefore $|H'| = |Q_1| \leqslant k/2$, which contradicts Lemma 1. Thus $Q \in A_2$ holds.

Suppose that $Q' \in A_1$. Since $Q \in A_2$, Q has a k-contractible edge e. By the symmetry of the roles of H' and W', we may assume that $V(e) \subset Q_1 \cup Q_2$. First suppose that $H_1 \neq \emptyset$. Then by the minimality of $|H|$, we have $|H_2 \cup Q_2 \cup Q_1| \geqslant k+2$, which implies $W_3 = \emptyset$, which contradicts Lemma 2(i). Thus we have $H_1 = \emptyset$. Next suppose that $H_3 \neq \emptyset$. By the minimality of $|H|$, we have $|H_2 \cup Q_2 \cup Q_3| \geqslant k+1$ because $H_2 \cup Q_2 \cup Q_3 \notin A_1$. Since e is not contained in any k-cutset, this forces $W_1 = \emptyset$ because now we have $|Q_1 \cup Q_2 \cup W_2| \leqslant k$. This contradicts Lemma 2(ii). So we have $H_1 = H_3 = \emptyset$. Then $|H_2| \geqslant (k+1)/2$.

Suppose that $W_1 \neq \emptyset$. Since e is not contained in any k-cutset, we have $|W_2 \cup Q_1 \cup Q_2| \geqslant k+1$. This implies that $|Q_1| \geqslant (k+3)/2$, and hence $|Q_2 \cup Q_3 \cup W_2| \leqslant k-1$. This forces $W_3 = \emptyset$. Then by the minimality of $|H|$, we have $|Q_3| \geqslant (k+1)/2$. Then $|Q| \geqslant |Q_1| + |Q_3| \geqslant k+2$, a contradiction. Thus we have $W_1 = \emptyset$. Suppose now that $W_3 \neq \emptyset$. Then by the minimality of $|H|$, we have $|Q_1| \geqslant (k+1)/2$. Since $|Q_2 \cup W_2| \leqslant (k-1)/2$, it follows that $|Q_2 \cup Q_3 \cup W_2| \leqslant k-1$, and this forces $W_3 = \emptyset$, a contradiction. Hence $W_3 = \emptyset$. Then by the minimality of $|H|$, now we have $|H_2| \geqslant (k+1)/2$, $|W_2| \geqslant (k+1)/2$. Consequently, $k+1 \leqslant |H_2| + |W_2| \leqslant |Q'| = k$, a contradiction. Hence $Q' \in A_2$ holds. □

In the rest of the proof, by Lemma 3, we may assume that for any edge $e \in E(H) \cup E(H, Q)$, $e \in E_c(G)$ and hence e is contained in a $(k+1)$-cutset $S \in A_2$. Again, in view of Lemma 3, we may assume that Q' is always chosen so that Q' contains a k-contractible edge in H(i.e., $|H_2| \geqslant 2$ holds.) Also, let f be a k-contractible edge in Q and fix it. Then f is contained in either $Q_1 \cup Q_2$ or $Q_2 \cup Q_3$. Without loss of generality, we may assume that f is contained in $Q_1 \cup Q_2$. We prove the following lemmas.

Lemma 4. $|H| \geqslant \lfloor k/2 \rfloor + 2$.

Proof. Suppose that $|H| = \lfloor k/2 \rfloor + 1$. Since now we have $E(H) \cup E(H, Q) \subset E_c(G)$ and $|Q| = k+1$, it follows from $\delta(G) \geqslant \lfloor 3k/2 \rfloor + 1$ that $H \cup Q$ contains a subgraph X such that $X \cong K_{\lfloor k/2 \rfloor + 1} + (k+1)K_1$ and $E(X) \subset E_c(G)$. Also, it is easy to see that X contains a vertex of degree exactly $\lfloor 3k/2 \rfloor + 1$ in G. $\square$

Lemma 5. *$H_3 = \emptyset$ and $H_1 \neq \emptyset$.*

Proof. First assume $H_3 \neq \emptyset$. Then by the minimality of H, $|H_2 \cup Q_2 \cup Q_3| \geqslant k+2$. If $W_1 \neq \emptyset$, then, since $Q_1 \cup Q_2 \cup W_2$ is a cutset containing f, $|Q_1 \cup Q_2 \cup W_2| \geqslant k+1$. But $2k+2 = \sum_{i=1}^{3} |Q_i| + |W_2| + |Q_2| + |H_2| = |H_2 \cup Q_2 \cup Q_3| + |Q_1 \cup Q_2 \cup W_2| \geqslant k+1+k+2 = 2k+3$, a contradiction. So, $W_1 = \emptyset$. This contradicts Lemma 2(ii). Thus $H_3 = \emptyset$ holds. Next assume that $H_1 = \emptyset$. Since now $H_3 = \emptyset$, $|H_2| \geqslant (k+3)/2$ by Lemma 4. This means that $|Q_2 \cup W_2| \leqslant (k-1)/2$. Since either $|Q_1| \leqslant (k+1)/2$ or $|Q_3| \leqslant (k+1)/2$ holds, we see that $W_1 = \emptyset$ or $W_3 = \emptyset$ holds because $Q_1 \cup Q_2$ contains a k-contractible edge.

First we consider the case where $W_1 = \emptyset$. Then, by the minimality of $|H|$, it follows that $|Q_1| \geqslant (k+3)/2$. This implies $W_3 = \emptyset$. Consequently, we have $|W'| = |Q_2| \leqslant (k-1)/2$, a contradiction. Arguing similarly in the case where $W_3 = \emptyset$, we can easily obtain a contradiction. $\square$

Lemma 6. $|Q_1 \cup Q_2 \cup H_2| = k+2$ *and* $|W_2 \cup Q_2 \cup Q_3| = k$.

Proof. By Lemma 5, now we have $H_1 \neq \emptyset$ and $H_3 = \emptyset$. Then by Lemma 2(i), $W_3 \neq \emptyset$. Hence by the minimality of $|H|$, $|Q_1 \cup Q_2 \cup H_2| \geqslant k+2$. Since G is k-connected, $|W_2 \cup Q_2 \cup Q_3| \geqslant k$. But $2k+2 = \sum_{i=1}^{3} |Q_i| + |W_2| + |Q_2| + |H_2| = |Q_1 \cup Q_2 \cup H_2| + |W_2 \cup Q_2 \cup Q_3|$, hence the equalities hold. $\square$

Since $|Q_1 \cup Q_2 \cup Q_3| = k+1$, by Lemma 6, we have $|H_2| = |Q_3| + 1$. Also, since $|H_2| \geqslant 2$, we have $|Q_3| \geqslant 1$. Next, we prove the following lemmas.

Lemma 7. $|N(U) \cap H| \geqslant |U| + 1$ *for all nonempty subsets U of $Q - V(f)$ with $H - N(U) \neq \emptyset$.*

Proof. Suppose there exists a nonempty subset U of $Q - V(f)$ with $|N(U) \cap H| \leqslant |U|$. Since $H - N(U) \neq \emptyset$, $(Q - U) \cup (N(U) \cap H)$ is a $(k+1)$-cutset containing $V(f)$ and separating $H - N(U)$ from $W \cup U$. But, since $|H - N(U)| < |H|$, this contradicts the minimality of $|H|$. $\square$

By Lemma 7, since $N(Q_3) \cap H \subset H_2$ and $|H_2| = |Q_3| + 1$, we have $N(Q_3) \cap H = H_2$. By Lemmas 3–7, we can obtain the following fact:

Put $E(H) = \{e_1, \ldots, e_{|E(H)|}\}$. For each edge $e_i \in E(H)$ with $1 \leqslant i \leqslant |E(H)|$, e_i is contained in some $(k+1)$-cutset $Q^i \in A_2$. Let H^i be some components in $G - Q^i$ such that $G - Q^i -$

$H^i \neq \emptyset$ and also, let W^i denote $G - Q^i - H^i$. Let H_1^i, H_2^i and H_3^i denote $H \cap H^i$, $H \cap Q^i$ and $H \cap W^i$, respectively. Also, let W_1^i, W_2^i and W_3^i denote $W \cap H^i$, $W \cap Q^i$ and $W \cap W^i$, respectively. Let Q_1^i, Q_2^i and Q_3^i denote $Q \cap H^i$, $Q \cap Q^i$ and $Q \cap W^i$, respectively. We may assume $V(f) \in Q_1^i \cup Q_2^i$. Then $H_3^i = \emptyset$, $|H_2^i| = |Q_3^i| + 1$ and $N(Q_3^i) \cap H = H_2^i$.

We prove the following lemma.

Lemma 8. *For each j with $2 \leqslant j \leqslant |E(H)|$, if $(N(Q_3^h) \cap H) \cap (\bigcup_{i=1}^{h-1} N(Q_3^i) \cap H) \neq \emptyset$ for every $2 \leqslant h \leqslant j$, then $|\bigcup_{i=1}^{j} N(Q_3^i) \cap H| \leqslant |\bigcup_{i=1}^{j} Q_3^i| + 1$.*

Proof. We prove by induction on j. Suppose $j = 2$. Since $N(Q_3^1) \cap H = H_2^1$ and $N(Q_3^2) \cap H = H_2^2$, we may assume that $H_2^1 \cap H_2^2 \neq \emptyset$. If $Q_3^1 \cap Q_3^2 = \emptyset$, then $|(N(Q_3^1) \cup N(Q_3^2)) \cap H| \leqslant |H_2^1 \cup H_2^2| \leqslant |H_2^1| + |H_2^2| - |H_2^1 \cap H_2^2| \leqslant |Q_3^1| + 1 + |Q_3^2| + 1 - 1 = |Q_3^1 \cup Q_3^2| + 1$. If $Q_3^1 \cap Q_3^2 \neq \emptyset$, then by Lemma 7, we have $|N(Q_3^1 \cap Q_3^2) \cap H| \geqslant |Q_3^1 \cap Q_3^2| + 1$, and hence $|(N(Q_3^1) \cup N(Q_3^2)) \cap H| \leqslant |H_2^1| + |H_2^2| - (|Q_3^1 \cap Q_3^2| + 1) = |Q_3^1| + 1 + |Q_3^2| + 1 - (|Q_3^1 \cap Q_3^2| + 1) = |Q_3^1 \cup Q_3^2| + 1$. Thus the result follows.

Assume $j \geqslant 3$. If $Q_3^j \subset \bigcup_{i=1}^{j-1} Q_3^i$, the result follows by the induction hypothesis. Assume $Q_3^j \not\subset \bigcup_{i=1}^{j-1} Q_3^i$, and let $R = Q_3^j \cap \bigcup_{i=1}^{j-1} Q_3^i$ and $S = (N(Q_3^j) \cap H) \cap (\bigcup_{i=1}^{j-1} N(Q_3^i) \cap H)$. Then $|S| \geqslant |R| + 1$ by the assumption of the lemma or by Lemma 7 according as $R = \emptyset$ or $R \neq \emptyset$. Hence we have $|\bigcup_{i=1}^{j} N(Q_3^i) \cap H| \leqslant |\bigcup_{i=1}^{j-1} Q_3^i| + 1 + |Q_3^j| + 1 - |R| - 1 = |\bigcup_{i=1}^{j} Q_3^i| + 1$. □

Lemma 9. $|H| \leqslant k$.

Proof. We define the following procedure. Let e_1 be any edge in H. Then e_1 is contained in some $(k+1)$-cutest $A^1 \in A_2$. Hence we can define Q_3^1 as in the above definition. Assume that we have defined e_l and $A^l \in A_2$ for $1 \leqslant l \leqslant j$. Hence we have already defined Q_3^l as in the above definition for $1 \leqslant l \leqslant j$. If $H - \bigcup_{i=1}^{j} N(Q_3^i) = \emptyset$, then we shall terminate this procedure. Otherwise, $H - \bigcup_{i=1}^{j} N(Q_3^i) \neq \emptyset$. Since H is connected, there exists an edge e_{j+1} joining from $\bigcup_{i=1}^{j} N(Q_3^i) \cap H$ to $H - \bigcup_{i=1}^{j} N(Q_3^i)$. Also, e_{j+1} is contained in some $(k+1)$-cutset $A^{j+1} \in A_2$.

When the procedure is terminated, we have $\bigcup_{i=1}^{|E(H)|} N(Q_3^i) \cap H = H$. Let e_{j_0} denote the last edge chosen in the procedure, and apply Lemma 8 with $j = j_0$. Then we have $|H| \leqslant |\bigcup_{i=1}^{|E(H)|} Q_3^i| + 1 \leqslant |Q - V(f)| + 1 \leqslant k$. □

Let $\mathcal{G}$ be a graph with $V(\mathcal{G}) = E(H)$ and $E(\mathcal{G}) = \{e_i e_j \mid i \neq j, N(Q_3^i) \cap N(Q_3^j) \neq \emptyset\}$. Now choose the subgraph induced by M in $\mathcal{G}$ with $\bigcup_{e_j \in M} N(Q_3^j) \cap H \neq H$ so that M is connected and $|(\bigcup_{e_j \in M} N(Q_3^j)) \cap H|$ is maximum, and subject to the condition that $|(\bigcup_{e_j \in M} N(Q_3^j)) \cap H|$ is maximum, $|M|$ is minimum. We may assume that $V(M) = \{e_1, \ldots, e_p\}$ where $p \leqslant |E(H)|$. Put $I = \bigcup_{1 \leqslant i \leqslant p} Q_3^i$.

Lemma 10. $|I| \geqslant (|H| - 1)/2$ *and the equality holds only if $|N(I) \cap H| = |N(Q_3^{p+1}) \cap H|$ and $|(N(I) \cap H) \cap (N(Q_3^{p+1}) \cap H)| = 1$.*

Proof. First we claim that $E(\bigcup_{e \in M} V(e), H - \bigcup_{e_j \in M} N(Q_3^j)) \neq \emptyset$. Suppose not, and take $x \in \bigcup_{e \in M} V(e)$, $y \in H - \bigcup_{e_j \in M} N(Q_3^j)$. Then it follows that $\lfloor 3k/2 \rfloor + 1 + 1 \leqslant |N_G(x) \cup \{x\}| \leqslant$

$|\bigcup_{e_j\in M} N(Q_3^j)\cap H| + |Q|$, and hence $\lfloor 3k/2\rfloor - k + 1 \leqslant |\bigcup_{e_j\in M} N(Q_3^j)\cap H|$. On the other hand, by Lemma 9, we have $\lfloor 3k/2\rfloor + 1 + 1 \leqslant |N_G(y)\cup\{y\}| \leqslant |H - \bigcup_{e_j\in M} N(Q_3^j)| + |\bigcup_{e_j\in M} N(Q_3^j)\cap H| - |\bigcup_{e\in M} V(e)| + |Q| - |I| = |H| - |\bigcup_{e\in M} V(e)| + k + 1 - |I| \leqslant 2k + 1 - |\bigcup_{e\in M} V(e)| - |I|$, and hence $|I| + 1 \leqslant 2k - \lfloor 3k/2\rfloor - |\bigcup_{e\in M} V(e)| < \lfloor 3k/2\rfloor - k + 1$ because $|\bigcup_{e\in M} V(e)| \geqslant 2$. Hence $|I| + 1 < |\bigcup_{e_j\in M} N(Q_3^j)\cap H|$. However, this contradicts Lemma 8. Thus $E(\bigcup_{e\in M} V(e), H - \bigcup_{e_j\in M} N(Q_3^j)) \neq \emptyset$ holds.

Take an edge e' in $E(\bigcup_{e\in M} V(e), H - \bigcup_{e_j\in M} N(Q_3^j))$. We may assume that $e' = e_{p+1}$. Then, by the maximality of $|(\bigcup_{e_j\in M} N(Q_3^j))\cap H|$, it follows that $N(I\cup Q_3^{p+1})\cap H = H$ and $|N(I)\cap H| \geqslant |N(Q_3^{p+1})\cap H|$ and, hence, arguing similarly in the proof of Lemma 9, we see that $|H| = |N(I\cup Q_3^{p+1})\cap H| = |N(I)\cap H| + |N(Q_3^{p+1})\cap H| - |(N(I)\cap H)\cap(N(Q_3^{p+1})\cap H)| \leqslant 2|N(I)\cap H| - 1 \leqslant 2|I| + 1$ by Lemma 8. □

Lemma 11. $|I| \leqslant |H| - \lfloor k/2\rfloor - 1$ *and the equality holds only if* $N_G(x)\cup\{x\} = H\cup(Q - I)$ *for every* $x\in H - N(I)$.

Proof. Take $x\in H - N(I)$. Then, since no vertex in I is adjacent to x, it follows that $(k+1) + |H| - |I| = |H\cup(Q - I)| \geqslant |N_G(x)\cup\{x\}| \geqslant \lfloor 3k/2\rfloor + 2$, as desired. □

By Lemmas 10 and 11, we have $|H| \geqslant 2\lfloor k/2\rfloor + 1$. In view of Lemma 9, it follows that k is odd and the equality holds. This means that $|H| = k$. Note that the equalities in both Lemma 10 and Lemma 11 hold. Hence, by the choice of M, it follows from Lemma 10 that $|M| = 1$ (i.e., $p = 1$). Also, by the symmetry of the roles of Q_3^1 and Q_3^2, we see from the equality in Lemma 11 that H is a complete graph of order k, $|Q_3^1| = |Q_3^2| = (k-1)/2$. Since $\delta(G) \geqslant \lfloor 3k/2\rfloor + 1$, this together with the above observation implies that $\langle H\cup(Q - Q_3^2)\rangle$ contains $K_{\lfloor k/2\rfloor+1} + (k+1)K_1$ each of whose edge is k-contractible. Moreover, there is a vertex in H that has degree exactly $\lfloor 3k/2\rfloor + 1$.

This is a contradiction. This completes the proof of Theorem 3. □

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